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A note on stability in three-phase-lag heat conduction

Ramón Quintanilla [a,*,1], Reinhard Racke [b,2]

[a] Department of Applied Mathematics II, UPC Terrassa, Colom 11, 08222 Terrassa, Spain
[b] Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany

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Abstract

In this note we consider two cases in the theory of the heat conduction models with three-phase-lag. For each one we propose a suitable Lyapunov function. These functions are relevant tools which allow to study several qualitative properties. We obtain conditions on the material parameters to guarantee the exponential stability of solutions. The spectral analysis complements the results and we show that if the conditions obtained to prove the exponential stability are not satisfied, then we can obtain the instability of solutions for suitable domains. We believe that this kind of results is fundamental to clarify the applicability of the models.

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1. Introduction

There are several parabolic and hyperbolic theories which describe the heat conduction, the latter also being called theories of second sound, where the propagation of heat is modelled with finite propagation speed, in contrast to the classical model using Fourier's law leading to infinite propagation speed of heat signals, see the surveys by Chandrasekharaiah [2] or Hetnarski and Ignaczak [9,10] or the book of Müller and Ruggeri [12]. Recently, there have been considered the dual-phase-lag heat equations which were proposed by Tzou [19–22] and investigated by Quintanilla and Racke [13–17] and Wang et al. [23–25].

In Refs. [2,9] there are described several models for the conduction of heat in the thermomechanical context. We can recall the models proposed by Lord and Shulman [11], Green and Lindsay [4], Hetnarski and Ignaczak [9], Green and Naghdi [5–8]. It is worth noting that the model proposed by Tzou contains the theories of Lord and Shulman, Green and Lindsay as particular cases. When several orders of approximation are considered in Tzou's theory the classical theory of Cattaneo [1] is obtained. However, the theories proposed by Green and Naghdi [5,6] cannot be obtained from this point of view. Recently Roy [18] has proposed a theory with three-phase-lag which is able to contain all the previous theories at the same time. The basic equation is

$$q(P, t + \tau_q) = -(k\nabla T(P, t + \tau_T) + k^*\nabla v(P, t + \tau_v)).$$

Here q is the heat flux vector, T is the temperature and v is the thermal displacement that satisfies $\dot{v} = T$. It seems that the proposal of Roy was to establish a mathematical model that includes three-phase-lags in the heat flux vector, the temperature gradient and in the thermal displacement gra-

* Corresponding author.
E-mail addresses: ramon.quintanilla@upc.edu (R. Quintanilla), reinhard.racke@uni-konstanz.de (R. Racke).
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dient. For this model, we can consider several kind of Taylor approximations to recover the previously cited theories. In particular the models of Green and Naghdi are recovered.

We focus our attention to the two new equations which are the following:

$$\rho c_v \ddot{T} + \tau_q \rho c_v \dddot{T} = k^* \Delta T + \tau_v^* \Delta \dot{T} + k\tau_T \Delta \ddot{T} \tag{1.1}$$

and

$$\rho c_v \ddot{T} + \tau_q \rho c_v \dddot{T} + \frac{\tau_q^2}{2} \rho c_v \ddddot{T} = k^* \Delta T + \tau_v^* \Delta \dot{T} + k\tau_T \Delta \ddot{T}, \tag{1.2}$$

where the coefficients appearing are positive constant material parameters. In particular $\tau_v^* = k^* \tau_v + k$.

It is worth noting that in case that $k^* = 0$ we have $\tau_v^* = k$ and Eqs. (1.1) and (1.2) become

$$\rho c_v \ddot{T} + \tau_q \rho c_v \dddot{T} = \tau_v^* \Delta \dot{T} + k\tau_T \Delta \ddot{T} \tag{1.3}$$

and

$$\rho c_v \ddot{T} + \tau_q \rho c_v \dddot{T} + \frac{\tau_q^2}{2} \rho c_v \ddddot{T} = \tau_v^* \Delta \dot{T} + k\tau_T \Delta \ddot{T}. \tag{1.4}$$

These equations are not new. They are examples of the dual-phase-lag heat conduction equations proposed by Tzou [19,20]. This can be seen introducing $\theta = \dot{T}$. Then the equations become

$$\rho c_v \dot{\theta} + \tau_q \rho c_v \ddot{\theta} = k\Delta\theta + k\tau_T \Delta\dot{\theta} \tag{1.5}$$

and

$$\rho c_v \dot{\theta} + \tau_q \rho c_v \ddot{\theta} + \frac{\tau_q^2}{2} \rho c_v \dddot{\theta} = k\Delta\theta + k\tau_T \Delta\dot{\theta}. \tag{1.6}$$

We recall that the stability aspects concerning these equations were given in [15].

We observe that Eq. (1.2) is of wave equation type for $\ddot{T}$,

$$\frac{\partial^2}{\partial t^2}(\ddot{T}) - \frac{2k\tau_T}{\tau_q^2 \rho c_v} \Delta \ddot{T} = \text{l.o.t.}$$

with lower order terms on the right-hand side. The characteristic wave speeds are

$$S_{1,2} = \pm\sqrt{\frac{2k\tau_T}{\tau_q^2 \rho c_v}}.$$

Actually, in one space dimension, we can reformulate (1.2) as first-order system for

$$V := \left(T, \dot{T}, \ddot{T}, \dddot{T}, T_x, \dot{T}_x, \ddot{T}_x\right)', \quad \dot{V} + AV_x + BV = 0$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2k^*}{\tau_q^2 \rho c_v} & -\frac{2\tau_v^*}{\tau_q^2 \rho c_v} & -\frac{2k\tau_T}{\tau_q^2 \rho c_v} \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\tau_q^2} & \frac{2}{\tau_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues γ_{1-7} of A are all real,

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0, \quad \gamma_{6,7} = \pm\sqrt{\frac{2k\tau_T}{\tau_q^2 \rho c_v}}.$$

Thus, it is a hyperbolic system with finite propagation speed, and we recover with $\gamma_{6,7}$ the characteristic wave speeds $S_{1,2}$.

Eq. (1.1) is of heat equation, i.e. parabolic, type for $\ddot{T}$,

$$\frac{\partial}{\partial t}(\ddot{T}) - \frac{k\tau_T}{\tau_q \rho c_v} \Delta \ddot{T} = \text{l.o.t.}$$

hence infinite propagation speed is expected.

In this paper we propose a Lyapunov function for each Eqs. (1.1) and (1.2). These functions will be a powerful tool to study the qualitative aspects of the solutions of these equations. We also consider the spectral analysis of these equations. Both analysis aspects characterize suitable conditions on the constitutive parameters to guarantee the stability or the instability of solutions, respectively.

We believe that this kind of results is fundamental to clarify the applicability of the models. In fact, if a model does not guarantee the stability of solutions, it cannot be a good candidate to describe the heat conduction. For the exponential stability we remark that there are examples for which Fourier's law of heat conduction leads to exponential stability, while Cattaneo's law does not, cp. [3]. Thus, we believe that, our results are the first step to clarify before to develop any qualitative study in these theories.

The paper is organized as follows: In Section 2 we prove (exponential) stability for Eq. (1.1) whenever condition (2.1) holds. A relevant, tool, a Lyapunov function is proposed. In Section 3, we get the instability of solutions whenever (2.1) fails. In the same manner, Sections 4 and 5 are devoted to the discussion of Eq. (1.2) with respect to the condition (4.1). In Section 6, we summarize our results.

2. Stability of solutions: Eq. (1.1)

In this section we will prove the stability of solutions of Eq. (1.1) whenever we assume that the constitutive constants satisfy:

$$\tau_v^* > \tau_q k^*. \tag{2.1}$$

We will consider the problem determined by Eq. (1.1) in a bounded domain B smooth enough to guarantee the use of the divergence theorem. To determine a well posed problem we impose the initial conditions

$$T(\mathbf{x},0) = T_0(\mathbf{x}), \quad \dot{T}(\mathbf{x},0) = \theta_0(\mathbf{x}),$$
$$\ddot{T}(\mathbf{x},0) = \eta_0(\mathbf{x}), \qquad \mathbf{x} \in B, \tag{2.2}$$

and the homogeneous boundary conditions

$$T(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial B. \tag{2.3}$$

Now, we will prove the exponential decay of solutions of the problem determined by Eq. (1.1), the initial conditions (2.2) and the boundary conditions (2.3). We define an energy functional which is a useful tool to study our problem:

$$F(t) := \frac{1}{2}\int_B \Big(\rho c_v(\dot{T} + \tau_q \ddot{T})^2 + k^*|\nabla(T + \tau_q \dot{T})|^2 + \big(\tau_q\big(\tau_v^* - k^*\tau_q\big) + k\tau_T\big)|\nabla \dot{T}|^2\Big)\mathrm{d}v. \tag{2.4}$$

We have

$$F'(t) = -\int_B \nabla(\dot{T} + \tau_q\ddot{T})\cdot\big(\big(\tau_v^* - k^*\tau_q\big)\nabla\dot{T} + k\tau_T\nabla\ddot{T}\big)\mathrm{d}v + \big(\tau_q\big(\tau_v^* - k^*\tau_q\big) + k\tau_T\big)\int_B \nabla\dot{T}\cdot\nabla\ddot{T}\mathrm{d}v. \tag{2.5}$$

Thus, we obtain

$$F'(t) = -\int_B \Big(\big(\tau_v^* - k^*\tau_q\big)|\nabla\dot{T}|^2 + k\tau_T\tau_q|\nabla\ddot{T}|^2\Big)\mathrm{d}v \leqslant 0. \tag{2.6}$$

This inequality shows that the solutions of our problem are stable whenever we assume that the constitutive coefficients satisfy condition (2.1). However, we are going to see a stronger result. We will prove that the solutions decay in an exponential way.

First, we note that the function F defines an energy of the type of $|\nabla T|^2, |\nabla\dot{T}|^2, |\ddot{T}|^2$. Now, we define the function

$$G(t) := \int_B \left(\frac{\tau_v^*}{2}|\nabla T|^2 + k\tau_T\nabla T\cdot\nabla\dot{T} + \rho c_v T\dot{T} + \rho c_v\tau_q T\ddot{T}\right)\mathrm{d}v.$$

We have

$$G'(t) = -k^*\int_B |\nabla T|^2\mathrm{d}v + \int_B \Big(k\tau_T|\nabla\dot{T}|^2 + \rho c_v\dot{T}^2 + \rho c_v\tau_q\dot{T}\ddot{T}\Big)\mathrm{d}v.$$

Now, it is clear that if we take ϵ a small enough we can guarantee that $F(t) + \epsilon G(t)$ defines a equivalent norm to $F(t)$ and such that

$$\int_B \Big(\big(\tau_v^* - k^*\tau_q\big)|\nabla\dot{T}|^2 + k\tau_T\tau_q|\nabla\ddot{T}|^2\Big)\mathrm{d}v - \epsilon\int_B \Big(k\tau_T|\nabla\dot{T}|^2 + \rho c_v\dot{T}^2 + \rho c_v\tau_q\dot{T}\ddot{T}\Big)\mathrm{d}v$$

is equivalent to an energy of the type of

$$\int_B \Big(|\nabla\dot{T}|^2 + |\nabla\ddot{T}|^2\Big)\mathrm{d}v.$$

Thus, it is clear that

$$F'(t) + \epsilon G'(t) \leqslant -\kappa\int_B \Big(|\nabla T|^2 + |\nabla\dot{T}|^2\Big)\mathrm{d}v \leqslant -\kappa^*(F(t) + \epsilon G(t)),$$

where κ and κ^* are two calculable positive constants. Thus, it follows the existence of two positive constants C and ϑ such that

$$F(t) \leqslant CF(0)\exp(-\vartheta t) \tag{2.7}$$

for all $t \geqslant 0$. This inequality says that the solutions decay in an exponential way.

3. Spectral analysis for (1.1)

We have proved that whenever we assume that inequality (2.1) holds, then the solutions decay in a exponential way. In this section we will see that whenever this condition fails, then there exist domains B such that the problem has unstable solutions. To this end, let us consider functions of the form

$$T(x,t) = \exp(\omega t)\Phi(x), \tag{3.1}$$

where Φ satisfies

$$\Delta\Phi + \lambda\Phi = 0 \quad \text{in } B \quad \text{and} \quad \Phi = 0 \quad \text{on } \partial B.$$

If function (3.1) is a solution of (1.1) ω must satisfy the equation

$$\tau_q\rho c_v x^3 + (\rho c_v + k\tau_T\lambda_n)x^2 + \tau_v^*\lambda_n x + k^*\lambda_n = 0, \tag{3.2}$$

where $(\lambda_n)_n$ denote the eigenvalues of the negative Laplace operator for Dirichlet boundary conditions.

By the Hurwitz criterion, we know that all three roots of the polynomial

$$\beta^3 + l_1\beta^2 + l_2\beta + l_3 = 0$$

have negative real parts if and only if

$$l_j > 0, \quad j = 1,2,3, \qquad l_1 l_2 > l_3 \tag{3.3}$$

holds. For (3.2) this conditions turns into

$$(\rho c_v + k\tau_T\lambda_n)\tau_v^* > \tau_q\rho c_v k^*.$$

If this condition shall be satisfied uniformly in λ_n then we need that (2.1) holds. In fact, we can extend this condition a little bit. Let the eigenvalues be arranged increasingly with λ_1 denoting the smallest eigenvalue. Then, for the stability, it is sufficient that

$$\lambda_1 > \frac{\left(\tau_q k^* - \tau_v^*\right)\rho c_v}{kT_T}. \tag{3.4}$$

On the other hand if (2.1) does not hold, we can always select a domain in such a way that (3.4) does not hold either and then, there exist unstable solutions.

4. Stability of solutions: Eq. (1.2)

The aim of this section is to prove that the solutions of the problem determined by Eq. (1.2), the initial conditions (2.2) and the boundary conditions (2.3) are stable whenever

$$\frac{2k\tau_T}{\tau_q} > \tau_v^* > k^*\tau_q \tag{4.1}$$

holds.

Our approach will be similar to the one used in Section 2 for Eq. (1.1). However, here the problem has a greater complexity requiring more sophisticated functionals.

Let us define the energy functional

$$\begin{aligned} F(t) :=& \frac{1}{2}\int_B \left(\rho c_v\left(\dot{T} + \tau_q\ddot{T} + \frac{\tau_q^2}{2}\dddot{T}\right)^2 + k^*\left|\nabla\left(T + \tau_q\dot{T} + \frac{\tau_q^2}{2}\ddot{T}\right)\right|^2\right. \\ &+ \tau_q\left(\tau_v^* - k^*\tau_q\right)\left|\nabla\left(\dot{T} + \frac{\tau_q}{2}\ddot{T}\right)\right|^2 + \left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\left|\nabla\dot{T}\right|^2 \\ &\left. + \frac{\tau_q^2}{2}\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\left|\nabla\ddot{T}\right|^2\right)\mathrm{d}v. \end{aligned} \tag{4.2}$$

We have

$$\begin{aligned} F'(t) =& -\int_B \left(\nabla\left(\dot{T} + \tau_q\ddot{T} + \frac{\tau_q^2}{2}\dddot{T}\right)\right. \\ &\left.\cdot\left(\left(\tau_v^* - k^*\tau_q\right)\nabla\dot{T} + \left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\nabla\ddot{T}\right)\right)\mathrm{d}v \\ &+ \int_B \left(\tau_q\left(\tau_v^* - k^*\tau_q\right)\nabla\left(\dot{T} + \frac{\tau_q}{2}\ddot{T}\right)\cdot\nabla\left(\ddot{T} + \frac{\tau_q}{2}\dddot{T}\right)\right. \\ &\left. + \left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\nabla\dot{T}\cdot\nabla\ddot{T}\right)\mathrm{d}v \\ &+ \int_B \left(\frac{\tau_q^2}{2}\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\nabla\ddot{T}\cdot\nabla\dddot{T}\right)\mathrm{d}v. \end{aligned} \tag{4.3}$$

It follows that the following equalities

$$\begin{aligned} F'(t) =& -\left(\tau_v^* - k^*\tau_q\right)\int_B |\nabla\dot{T}|^2\mathrm{d}v - \left(\tau_q\left(k\tau_T - k^*\frac{\tau_q^2}{2}\right) - \frac{\tau_q^2}{2}\left(\tau_v^* - k^*\tau_q\right)\right)\int_B |\nabla\ddot{T}|^2\mathrm{d}v \\ &- \left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\int_B \nabla\dot{T}\cdot\nabla\ddot{T}\mathrm{d}v - \tau_q\left(\tau_v^* - k^*\tau_q\right)\int_B \nabla\ddot{T}\cdot\nabla\dot{T}\mathrm{d}v \\ &- \frac{\tau_q^2}{2}\left(\tau_v^* - k^*\tau_q\right)\int_B \nabla\dddot{T}\cdot\nabla\dot{T}\mathrm{d}v - \frac{\tau_q^2}{2}\left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\int_B \nabla\dddot{T}\cdot\nabla\ddot{T}\mathrm{d}v \\ &+ \tau_q\left(\tau_v^* - k^*\tau_q\right)\int_B \nabla\dot{T}\cdot\nabla\ddot{T}\mathrm{d}v + \tau_q\frac{\tau_q^2}{2}\left(\tau_v^* - k^*\tau_q\right)\int_B \nabla\dot{T}\cdot\nabla\dddot{T}\mathrm{d}v \\ &+ \tau_q\frac{\tau_q^2}{4}\left(\tau_v^* - k^*\tau_q\right)\int_B \nabla\ddot{T}\cdot\nabla\dddot{T}\mathrm{d}v + \left(k\tau_T - k^*\frac{\tau_q^2}{2}\right)\int_B \nabla\dot{T}\cdot\nabla\ddot{T}\mathrm{d}v \\ &+ \frac{\tau_q^2}{2}\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\int_B \nabla\ddot{T}\cdot\nabla\dddot{T}\mathrm{d}v \end{aligned} \tag{4.4}$$

$$\begin{aligned} =& -\left(\tau_v^* - k^*\tau_q\right)\int_B |\nabla\dot{T}|^2\mathrm{d}v - \tau_q\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\int_B |\nabla\ddot{T}|^2\mathrm{d}v \\ &+ \left(-\frac{\tau_q^2}{2}\left(k\tau_T - k^*\frac{\tau_q^2}{2}\right) + \frac{\tau_q^3}{4}\left(\tau_v^* - k^*\tau_q\right) + \frac{\tau_q^2}{2}\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\right)\int_B \nabla\ddot{T}\cdot\nabla\dddot{T}\mathrm{d}v \end{aligned} \tag{4.5}$$

hold. Thus, we obtain

$$\begin{aligned} F'(t) =& -\left(\tau_v^* - k^*\tau_q\right)\int_B |\nabla\dot{T}|^2\mathrm{d}v \\ &- \tau_q\left(k\tau_T - \frac{\tau_q\tau_v^*}{2}\right)\int_B |\nabla\ddot{T}|^2\mathrm{d}v. \end{aligned} \tag{4.6}$$

This inequality shows the stability of solutions whenever conditions (4.1) holds. Again, we want to prove the exponential decay. For this purpose we define the function

$$\begin{aligned} G_1(t) :=& \frac{1}{2}\int_B \left(\frac{\tau_q^2}{2}\rho c_v(\dddot{T})^2 + \rho c_v(\ddot{T})^2 + k\tau_T|\nabla\ddot{T}|^2\right. \\ &\left. + 2\tau_v^*\nabla\dot{T}\cdot\nabla\ddot{T} + 2k^*\nabla T\cdot\nabla\ddot{T}\right)\mathrm{d}v. \end{aligned}$$

We have

$$G_1'(t) = -\tau_q\rho c_v\int_B |\dddot{T}|^2\mathrm{d}v + \int_B \left(\tau_v^*|\nabla\ddot{T}|^2 + k^*\nabla\dot{T}\cdot\nabla\ddot{T}\right)\mathrm{d}v.$$

We also define a new function

$$\begin{aligned} G_2(t) :=& \int_B \left(\frac{\tau_v^*}{2}|\nabla T|^2 + k\tau_T\nabla T\cdot\nabla\dot{T}\right. \\ &\left. + \rho c_v T\dot{T} + \tau_q\rho c_v T\ddot{T} + \frac{\tau_q^2}{2}\rho c_v T\dddot{T}\right)\mathrm{d}v. \end{aligned}$$

We have

$$\begin{aligned} G_2'(t) =& -k^*\int_B |\nabla T|^2\mathrm{d}v \\ &+ \int_B \left(k\tau_T|\nabla\dot{T}|^2 + \rho c_v(\dot{T})^2 + \tau_q\rho c_v\dot{T}\ddot{T} + \frac{\tau_q^2}{2}\rho c_v\dot{T}\dddot{T}\right)\mathrm{d}v. \end{aligned}$$

If we take ϵ_2 small enough we can guarantee that

$$\begin{aligned} \epsilon_2 G_2'(t) + G_1'(t) \leqslant& -\epsilon_2 k^*\int_B |\nabla T|^2\mathrm{d}v - \int_B \frac{\tau_q\rho c_v}{2}|\dddot{T}|^2\mathrm{d}v \\ &+ C^*\int_B \left(|\nabla\dot{T}|^2 + |\nabla\ddot{T}|^2\right)\mathrm{d}v. \end{aligned} \tag{4.7}$$

Here C^* is a computable positive constant.

If we take now ϵ_1 small enough, we can guarantee that the function $F(t)+\epsilon_1(G_1(t)+\epsilon_2 G_2(t))$ is equivalent to the function $F(t)$. At the same time we can see that

$$\begin{aligned} F'(t)+\epsilon_1\left(G_1'(t)+\epsilon_2 G_2'(t)\right) &\leqslant -\kappa\int_B\left(|\nabla T|^2+|\nabla\dot{T}|^2+|\nabla\ddot{T}|^2+(\dddot{T})^2\right)\mathrm{d}v \\ &\leqslant -\kappa^*(F(t)+\epsilon_1(G_1(t)+\epsilon_2 G_2(t))). \end{aligned} \quad (4.8)$$

Again κ and κ^* are calculable positive constants. Thus, we can obtain an estimate similar to (2.7) and the exponential stability of solutions is proved.

5. Spectral analysis for (1.2)

The aim of this section is to prove that whenever condition (4.1) fails then there exist domains B such that unstable solutions of the problem determined by Eq. (1.2) and the boundary and initial conditions (2.2) and (2.3) exist.

If we look for solutions of the form (3.1) to Eq. (1.2), ω must satisfy the equation

$$\frac{\rho c_v \tau_q^2}{2}x^4+\rho c_v\tau_q x^3+(\rho c_v+k\tau_T\lambda_n)x^2+\tau_v^*\lambda_n x+k^*\lambda_n=0. \quad (5.1)$$

We can write it in the following form

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

where

$$a_1=\frac{2}{\tau_q},\quad a_2=\frac{2}{\tau_q^2}+\frac{2k\tau_T\lambda_n}{\tau_q^2\rho c_v},\quad a_3=\frac{2\tau_v^*\lambda_n}{\tau_q^2\rho c_v},\quad a_4=\frac{2k^*\lambda_n}{\tau_q^2\rho c_v}.$$

We know that the Hurwitz criterium says that the solutions of the equation have negative real part if and only if all the a_i are positive and the following conditions:

$$a_1a_2-a_3>0,\quad a_1a_2a_3-a_1^2a_4-a_3^2>0$$

hold. In our case, we have

$$\begin{aligned} a_1a_2-a_3&=\frac{2}{\tau_q}\left(\frac{2}{\tau_q^2}+\frac{2k\tau_T\lambda_n}{\tau_q^2\rho c_v}\right)-\frac{2\tau_v^*\lambda_n}{\tau_q^2\rho c_v}\\ &=\frac{4}{\tau_q^3}+\frac{4k\tau_T-2\tau_q\tau_v^*}{\tau_q^3\rho c_v}\lambda_n. \end{aligned}$$

To guarantee that this is positive for every λ_n, we need to impose

$$\tau_v^*<\frac{2k\tau_T}{\tau_q}. \quad (5.2)$$

The second relation is

$$\begin{aligned} a_1a_2a_3-a_1^2a_4-a_3^2&=\frac{2}{\tau_q}\left(\frac{2}{\tau_q^2}+\frac{2k\tau_T\lambda_n}{\tau_q^2\rho c_v}\right)\frac{2\tau_v^*\lambda_n}{\tau_q^2\rho c_v}\\ &\quad-\left(\frac{2}{\tau_q}\right)^2\frac{2k^*\lambda_n}{\tau_q^2\rho c_v}-\left(\frac{2\tau_v^*\lambda_n}{\tau_q^2\rho c_v}\right)^2\\ &=\frac{8\lambda_n}{\tau_q^5\rho c_v}\left(\tau_v^*-k^*\tau_q\right)\\ &\quad+\frac{\tau_v^*\lambda_n^2}{\tau_q^5\rho^2c_v^2}\left(8k\tau_T-4\tau_v^*\tau_q\right). \end{aligned}$$

If we assume that (5.2) holds, the second condition is

$$k^*\tau_q<\tau_v^*. \quad (5.3)$$

If we assume that (5.2) fails, we can always select λ_n large enough to guarantee that

$$\frac{4}{\tau_q^3}+\frac{4k\tau_T-2\tau_q\tau_v^*}{\tau_q^3\rho c_v}\lambda_n<0$$

and then, there exists instability of solutions.

In case that (5.2) holds, but (5.3) fails, we can always select a domain B with a geometry such that the first eigenvalue λ_1 is small enough to guarantee that

$$\frac{2}{\tau_q^5\rho c_v}\left(\tau_v^*-k^*\tau_q\right)+\frac{\tau_v^*\lambda_1}{\tau_q^5\rho^2c_v^2}\left(2k\tau_T-\tau_v^*\tau_q\right)<0. \quad (5.4)$$

Thus, we will obtain unstable solutions to our problem.

We also point out that in case that (5.2) holds and the geometry of B is good enough to guarantee that

$$\frac{2}{\tau_q^5\rho c_v}\left(\tau_v^*-k^*\tau_q\right)+\frac{\tau_v^*\lambda_1}{\tau_q^5\rho^2c_v^2}\left(2k\tau_T-\tau_v^*\tau_q\right)>0$$

then the solutions are stable.

6. Conclusion

In this short note we have analyzed the range of the parameters τ_T, τ_v, k, k^* and τ_q for the three-phase-lag theory in order to guarantee the that the solutions of the corresponding heat equation are stable. For Eq. (1.1) we have seen that

1.a. If $\tau_v^*-k^*\tau_q<0$, then, there may exist unstable solutions, depending on the domain.
1.b. If (3.4) is violated, there always exist unstable solutions.
2. If $\tau_v^*-k^*\tau_q>0$, then the solutions are always (exponentially) stable.
3. In the particular case that $k^*=0$, we recover the dual-phase-lag problem and the solutions are always stable. This agrees with the analysis developed in [15].
For Eq. (1.2) we have seen that
4. If $\tau_q\tau_v^*>2k\tau_T$ there always exist unstable solution.
5.a. If $\tau_q\tau_v^*<2k\tau_T$ but $\tau_v^*>k^*\tau_q$ then there may exist unstable solutions, depending on the domain.
5.b. If (5.4) holds, there always exist unstable solutions.
6. If $k^*\tau_q<\tau_v^*<\frac{2k\tau_T}{\tau_q}$, then the solutions are always (exponentially) stable.

7. In the particular case that $k^* = 0$ we have $\tau_v^* = k$ and we recover the dual-phase-lag problem. The solutions are stable (unstable) whenever $\tau_q < (>) 2\tau_T$. This agrees with the analysis developed in [15].

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